

Critical velocities and the effect of steady and oscillating rotations on solid He4

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Abstract

We apply our recently developed model of a Bose condensate of quantum kink wave in solid He4 to understand recent torsional oscillator experimental results of the critical velocities and the effect of the steady and oscillating rotations at around 0.1 degree K. When the D.C. rotation is present we find a decrease of the Q factor given by $Q^{-1} \propto f_{sf} \times \Omega_D / \omega_{TO}$ where f_{sf} is the superfluid fraction; Ω_D , the D. C. angular rotation velocity, ω_{TO} , the torsional oscillator oscillating frequency. We estimate the AC critical velocity Ω_A^{crit} as that required to generate a kink wave of wavevector $2\pi/L_d$ where L_d is the distance between nodes of the dislocation network. We generalize this to include a steady rotation and find a D. C. critical velocity $\Omega_D^{crit} \propto (\Omega_A^{crit})^{1/2}$. Estimates for both the steady and the oscillating critical velocities are in order of magnitude agreement with experimental results. We have also examined an alternative mechanism of kink tunnelling through a node in the dislocation network and find that there is also a dependence on the torsional oscillator frequency: $\Omega_D^{crit} = [\Omega_A^{crit} \omega_{TO} 2\pi]^{1/2}$. The DC critical velocity Ω_D^{crit} is ten times higher than the experimental value.

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Since the discovery of an increase (1 per cent) in the solid ^4He moment of inertia in torsional oscillator (TO) experiments at around 200 mK[1], there have been renewed interests in its low temperature physical properties[2–7]. Many novel physical behavior are manifested, such as a very small direct flow and a very small critical velocity ($\Omega_A^{crit} \approx 10^{-3}$ rad/s). Recently TO experiments are carried out in the presence of both a steady (DC) and an oscillating (AC) rotation[8, 9]. An increase in damping is observed which increases with the DC rotation speed. When the AC rotation velocity is below the critical value, there is also a DC critical velocity Ω_D^{crit} which is three orders of magnitude larger than the AC critical velocity. The DC rotation does not affect the shear modulus. The AC speed of the Kubota group[9], $60\mu\text{m}$, is higher than Ω_A^{crit} . No critical DC velocity was observed.

We have recently studied the physics of kink waves of dislocations of density n_d and their Bose-Einstein condensation (BEC) [10]. The BEC of the kinks makes possible dissipationless movement of the dislocation lines. The motion of a dislocation corresponds in part to a circular motion of many He4 atoms, each by a different amount. An estimate of the fraction of He4 atoms can be obtained by weighting with respect to the strains. With this the corresponding "superfluid fraction" due to the motion of the dislocation lines is found to be of the order of $n_d a_0 L_m$, a magnitude that is consistent with current experimental results. Here a_0 is the lattice constant, L_m is the mosaic size. The dislocation motion does not produce any net linear motion of the atoms and thus will not generate any direct superflow. In this paper we apply our model to understand the experimental results of the critical velocities and the effect of the DC and AC rotations. We estimate the AC critical velocity Ω_A^{crit} as that required to generate a kink wave of wavevector $2\pi/L_d$ where L_d is the distance between nodes of the dislocation network. When the DC rotation is present we find a decrease of the Q factor given by $Q^{-1} \propto f_{sf} \times \Omega_D/\omega_{TO}$ where f_{sf} is the superfluid fraction; Ω_D , the D. C. angular rotation velocity, ω_{TO} , the torsional oscillator oscillating frequency. We have examined two mechanisms for a DC critical velocity Ω_D^{crit} : (1) The DC rotation generates kinks with a time independent displacement. Oscillating kinks are in turn generated from this state by the oscillating rotation. (2) Similar to the Josephson effect, the combination of the DC and AC rotation can cause a steady current of kinks across nodes of the dislocation network when the DC rotation is fast enough. We find that for both mechanisms $\Omega_D^{crit} \propto (\Omega_A^{crit})^{1/2}$ where Ω_A^{crit} is the critical AC angular velocity. For the second mechanism, there is also a dependence on the torsional oscillator frequency: $\Omega_D^{crit} = [\Omega_A^{crit} \omega_{TO} 2\pi]^{1/2}$. Using current

experimental estimates for the different physical parameters, we find Ω_A^{crit} of the same order of magnitude as the experimental value. An estimate of the DC critical velocity Ω_D^{crit} with the first mechanism is of the same order of magnitude as the experimental results; with the second mechanism the critical velocity is ten times higher than the experimental value. We hope this paper will stimulate further experiments and provide tests of the validity of our picture. We now describe our results in detail.

As is well known[11], for a body of density ρ rotating with angular frequency Ω , the quantity of interest is

$$F = E - \Omega \cdot \mathbf{M} \quad (1)$$

where E , \mathbf{M} are the energy and the angular momentum measured with respect to a coordinate system fixed in space. For example, for a simple rotation at frequency Ω , consider a body rotating at velocity ν . Then $E(\nu) = 0.5 \int d\mathbf{r} \rho \nu^2 r^2$ and $M = \nu \int d\mathbf{r} \rho r^2$. Minimizing F with respect to ν we get $\nu = \Omega$, as we expected. At this frequency $F(\nu = \Omega) = F_0 = -0.5\Omega^2 \int d\mathbf{r} \rho r^2$

In the presence of only a time dependent oscillating rotation with an angular frequency $\Omega_{TO}(t) = \Omega_A \exp(i\omega_{TO}t)$ caused by a torsional oscillator, if the dislocations also move with the entire solid, the energy of the system would be $E_0 = 0.5 \int d\mathbf{r} \rho [\Omega_{TO}(t) \times (\mathbf{r} + \mathbf{u})]^2$. Here \mathbf{u} is the displacement due to the dislocations. In our picture, as the He4 is rotated, the kinks remain in the zero momentum condensate relative to a space fixed coordinate system. This reduces the kinetic energy of the system. We obtain $E' = 0.5 \int d\mathbf{r} \rho [\Omega_{TO}(t) \times (\mathbf{r} + \mathbf{u} + \Delta\mathbf{u})]^2$. Here $\Omega_{TO}(t) \times \Delta\mathbf{u}$ is the reduction in the angular velocity where $\Delta\mathbf{u}$ comes from the motion of the dislocations relative to the rotating solid. The "superfluid fraction" is given by $f_{sf} = (E_0 - E')/E_0$.

We first estimate the critical velocity when only the time dependent oscillation with angular velocity $\Omega_{TO}(t)$ is present. We consider that a critical velocity is reached when it becomes possible to excite a kink wave so that it is possible to lower F ; ΔF becomes negative. For a network of dislocations the lowest wavevector is of the order of $k_0 = 2\pi/L_d$ where L_d is the distance between nodes. This wave vector can be further increased if the dislocation moves close to defects (He3) which provide further pinning. In that case, the critical velocity will become higher. We think a lot of the recently observed hysteretic behaviour[12] is related to this issue.

An example of a kink wave oscillating with frequency ω_{TO} is given by:

$$|\psi\rangle = \sin \omega_{TO} t / 2 | -k_0 \rangle + \cos \omega_{TO} t / 2 | k_0 \rangle . \quad (2)$$

The average velocity of this state is given by

$$v = \langle \psi | \hat{v} | \psi \rangle = v_0 \cos \omega_{TO} t \quad (3)$$

where

$$v_0 = \hbar k_0 / m^*, \quad (4)$$

m^* is the effective mass of the kink wave. We next proceed to estimate ΔF .

If there are N kinks per unit length, each with velocity v the velocity at which the dislocation moves out can be estimated as follows. For the dislocation to move out by a lattice constant a , each kink has to move a distance of $1/N$. The time it takes to do this is $\Delta t = 1/(Nv)$. Denoting the position of a dislocation by c , the speed of the dislocation due to the finite speed of the kink is

$$\partial c / \partial t = a \exp(i\omega_{TO} t) / \Delta t = a \exp(i\omega_{TO} t) N v. \quad (5)$$

A dislocation with the Burger's vector along the x direction at some point (c_x, c_y) along the z axis causes an atom at position (x, y) to move by $u_x(\mathbf{r} - \mathbf{c})$, $u_y(\mathbf{r} - \mathbf{c})$ [14]. The atomic displacement depends on the position of the dislocation.

$$\partial \mathbf{u} / \partial t = -\nabla_c \mathbf{u} \cdot \partial \mathbf{c} / \partial t. \quad (6)$$

is the corresponding velocity of an atom a distance \mathbf{r}' away from the moving dislocation due to the finite speed of the kinks. In the following, we shall assume that the dislocation moves in the direction along the Burger's vector which we take to be the x axis. In general, the axis of rotation is not parallel to the axis of the dislocation. With respect to the rotation axis, the actual displacement should be $\mathbf{R} \mathbf{u} (\mathbf{R}^{-1} \mathbf{r})$ where \mathbf{R} is the rotation matrix that can be specified by the Euler angles. We shall assume that this is the case and for simplicity of notation, not displayed this dependence at every step.

When the kink wave is created, the kinetic energy cost for a segment of the dislocation between nodes is given by [13]

$$\Delta E = N L_d (\hbar^2 k_0^2 / 2 m^*) \quad (7)$$

where $\delta \mathbf{u}$ is the displacement caused by this state. The term $\int d\mathbf{r} \rho (\partial u / \partial t)^2$ has already been included in the kinetic energy of the kink[15] and thus need not be counted twice.

The change in the angular momentum due to a change of the state ($r \gg u$) of the kinks is given by

$$\Delta \mathbf{M} \approx \int d^2 r \rho \mathbf{r} \times \partial \mathbf{u}(\mathbf{r}') / \partial t$$

There is another term $\int d^2 r \rho \mathbf{r} \times (\boldsymbol{\Omega}_{\text{TO}}(\mathbf{t}) \times \delta \mathbf{u})$ which provides a zero time average to $\Omega_{TO}(t) \Delta M$ and thus will be ignored from now on. From eqs. (5) and (6) $\Delta \mathbf{M}$ is of the order

$$\Delta \mathbf{M} \approx \int d^2 r \rho \mathbf{r} \times am N v L_d \partial_{x'} \mathbf{u}(\mathbf{r}') \quad (8)$$

In general \mathbf{u} is a sum of contributions from different dislocations located at different positions \mathbf{c}_i : $\mathbf{u} = \sum_i \mathbf{u}_0(\mathbf{r}' - \mathbf{c}_i)$. $\Delta \mathbf{M}$ can be written as a sum of contributions from each of the dislocations.

$$\Delta \mathbf{M} \approx \sum_i \int d^2 r \rho (\mathbf{r} - \mathbf{c}_i) \times am N v L_d \partial_{x'} \mathbf{u}(\mathbf{r}' - \mathbf{c}_i) \quad (9)$$

The range of integration of each of these terms is of the order of the mosaic size. Since $\partial_{x'} \mathbf{u}(\mathbf{r}' - \mathbf{c}_i)$ is of the order of $1/|r' - c_i|$ for a single dislocation, we obtain

$$\Delta \mathbf{M} \approx 2\pi m^* N v L_d L_m^2 / a \quad (10)$$

where L_m is the mosaic size. From eq. (7), (1) and the condition that $\Delta F = 0$, we get the critical angular velocity

$$\Omega_A^{crit} = \Delta E / \Delta M \approx \hbar a / (2\pi m^* L_d L_m^2). \quad (11)$$

Using experimental estimates of $L_d = 5\mu m$, $L_m = 20\mu m$, and our estimate $m^* \approx 0.1 m_{He4}$. We get $\Omega_A^{crit} \approx 10^{-3}/s$, of the same order of magnitude as the experimental results.

We next consider the case where the solid is rotating with a constant angular velocity Ω_D and ask if it is energetically favorable to start moving the kinks to a state of finite momentum. Instead of an "oscillating state" as in eq. (2), we consider the possibility of creating simple states $|\pm k_0\rangle$. The velocity of the kinks will then just be v_0 instead of v . Going through the same algebra, we arrive at a DC critical velocity that is the same order of magnitude as the AC critical velocity. The experimental DC angular velocity is higher than the AC angular velocity by two orders of magnitude. We thus assume a state so that the dislocations move with the entire solid with the constant angular velocity Ω_D .

In the additional presence of an oscillating driving term so that the total angular velocity is $\Omega = \Omega_D + \Omega_{TO}(t)$, we now consider if the dislocations will exhibit oscillating movements.

If the kinks do not exhibit the oscillating motion, the kinetic energy saved is given by

$$\Delta E = \int d\mathbf{r} \rho \mathbf{v}_0 \cdot \delta \mathbf{v}.$$

Here $\mathbf{v}_0 = [\boldsymbol{\Omega}_D + \boldsymbol{\Omega}_{TO}(\mathbf{t})] \times \mathbf{r}$ is the velocity of the solid in constant rotation. $\delta \mathbf{v}(t)$ is the change in velocity due to the kinks not moving with an oscillating velocity so that the core position $\delta c(t)$ exhibits a oscillating time dependence relative to the rotating solid. The change in velocity now has an additional contribution from the coupling of the steady rotation:

$$\delta \mathbf{v} = -\nabla_c \mathbf{u} \cdot (\Omega_D \times \Delta c + \partial \Delta \mathbf{c} / \partial t). \quad (12)$$

$\partial \Delta \mathbf{c} / \partial t \approx -\boldsymbol{\Omega}_{TO}(t) \times \mathbf{c}$, $\Delta \mathbf{c} \approx -\boldsymbol{\Omega}_{TO}(t) \times \mathbf{c} / i\omega_{TO}$. Because \mathbf{v}_0 is a sum of two terms, $\Delta E = \Delta E_D + \Delta E_A$ contains two contributions: those from coupling to Ω_D and those from coupling to $\Omega_{TO}(t)$. The coupling term to the **constant** DC rotation is given by

$$\Delta E_D = \int d\mathbf{r} \rho \boldsymbol{\Omega}_D \times \mathbf{r} \cdot \delta \mathbf{v}(t)$$

which has a zero time average. Thus the DC rotation cannot directly drive the dislocations to a finite oscillating velocity. The coupling term to the **oscillating** rotation is given by

$$\Delta E_A = \int d\mathbf{r} \rho \boldsymbol{\Omega}_{TO}(\mathbf{t}) \times \mathbf{r} \cdot \delta \mathbf{v}(t),$$

Because δv is a sum of two terms (eq. 12), $\Delta E_A = \Delta E_{A1} + \Delta E_{A2}$ where $\Delta E_{A1} = \int d\mathbf{r} \rho \boldsymbol{\Omega}_{TO}(\mathbf{t}) \times \mathbf{r} (\nabla_c \mathbf{u} \cdot \partial \Delta \mathbf{c} / \partial t)$, $\Delta E_{A2} = \int d\mathbf{r} \rho \boldsymbol{\Omega}_{TO}(\mathbf{t}) \times \mathbf{r} (\nabla_c \mathbf{u} \cdot \Omega_D \times \Delta c)$. ΔE_A has a nonzero time average. The ratio $\Delta E_{A1} / E \approx f_{sf}$ provides for the effective reduction of the moment of inertia and is of the order of the "superfluid fraction" f_{sf} . Now $\Delta \mathbf{c}$ and $\partial \Delta \mathbf{c} / \partial t$ in eq. (12) and hence ΔE_{A1} and ΔE_{A2} are ninety degree out of phase in time. We thus expect ΔE_{A2} to provide for a damping term, as is observed in the experiments. The ratio $\Delta c / (\partial \Delta c / \partial t)$ is of the order of $1 / \omega_{TO}$, the inverse torsional oscillator vibration frequency. The total rotation energy E_T of the system is a sum of the rotation energy of the container and that of solid He4, E_0 . The Q factor is defined with respect to E_T . We write $E_0 = \alpha E_T$ for a constant α . We thus expect the ΔE_{A2} term to provide a damping that is of the order of E_T / Q where

$$1/Q \approx \alpha f_{sf} \Omega_D / \omega_{TO}. \quad (13)$$

Taking a superfluid fraction of the order of 1 per cent, a Ω_D of the order of 1 rad/s and $\omega_{TO} = 2\pi \times 10^3 \text{ rad/s}$, we obtain an estimate of Q that is of the order of $10^6 \alpha$. Experimentally, Q^{-1} ranges from 10^{-6} to 10^{-9} . Our estimate is consistent with this. $1/Q$ scales with Ω_D and f_{sf} , also consistent with experimental findings. We next examine the critical DC rotation field. We have considered two possible mechanisms. We describe them sequentially next.

(i) We again examine the energetics of creating a kink wave of wavevector $2\pi/L_d$. Before the kink wave is created, the atoms are at positions $r_i + u_i + \Delta u_i$. Because $\Delta u_i \ll u_i$, we shall neglect the contribution due to Δu_i below. The angular momentum is now given by

$$M = m \sum_i [\mathbf{r}_i + \mathbf{u}_i + \delta \mathbf{u}_i(t)] \times ([\boldsymbol{\Omega}_D + \boldsymbol{\Omega}_{TO}(t)] \times [\mathbf{r}_i + \mathbf{u}_i + \delta \mathbf{u}_i(t)] + \partial \mathbf{u}_i / \partial t)$$

The velocity \mathbf{v}_i is a sum of that due to motion of the kink, $\partial u_i / \partial t$, and that due to the rotation $\Omega_D + \Omega_{TO}$. The corresponding energy is

$$E = 0.5m \sum_i ([\boldsymbol{\Omega}_D + \boldsymbol{\Omega}_{TO}(t)] \times [\mathbf{r}_i + \mathbf{u}_i + \delta \mathbf{u}_i(t)])^2 + NL_d \hbar^2 / (2mL_d^2).$$

Recall that before the kink wave is created, the energy is

$$E_0 = 0.5m \sum_i ([\boldsymbol{\Omega}_D + \boldsymbol{\Omega}_{TO}(t)] \times [\mathbf{r}_i + \mathbf{u}_i])^2.$$

The change in energy is thus

$$\Delta E \approx NL_d \hbar^2 / (2m_{kink} L_d^2) + m \sum_i (\Omega_D + \Omega_{TO})^2 [0.5 \delta u_i(t)^2 + \delta u_i(t)(r_i + u_i)].$$

We now look at ΔF , the change in F as a kink wave is created.

In general, $r \gg u(r)$, after discarding contributions with zero time averages, we obtain

$$\Delta F \approx NL_d \hbar^2 / (2m_{kink} L_d^2) - m \sum_i [\Omega_D^2 \delta u_i(t) r_i + 2\Omega_{TO}(t) \Omega_D r_i \delta u_i(t) + \Omega_{TO}(t) r_i \partial u_i / \partial t] \quad (14)$$

The last term is the same as in the AC case. Since $|\Omega_D| \gg |\Omega_A|$, the term $2\Omega_{TO}(t) \Omega_D r_i \delta u_i(t)$ is much smaller than $\Omega_D^2 \delta u_i(t) r_i$ and will be ignored. In this sum there is now a new driving term $-m \sum_i \Omega_D^2 \delta u_i(t) r_i$ that couples to a constant change of position of the kinks. Consider, for example, the wave function $\phi(z) \propto [1 + \sin(2\pi z/L_d)]$ which is a linear combination of the state $|k=0\rangle$ and the states $|k=\pm 2\pi/L_d\rangle$. This state has a constant shift in the kink position. Once this state is created, the oscillating Hamiltonian can couple the states ϕ to an oscillating state such as $\phi'(z) \propto \cos \omega_{TO} [1 + \sin(2\pi z/L_d)]$.

The displacement δu_i is of the order L_d , the new term is of the order of magnitude $-m\Omega_D^2 L_d L_m^2/a^2$. Substituting this into eq. (14) and setting $\Delta F = 0$ we thus arrive at a critical DC angular velocity of the order of magnitude

$$\Omega_D^{crit} \approx [\Omega_A^{crit} v_0/L_d]^{1/2} \quad (15)$$

From this we obtain an estimate of Ω_D^{crit} of the order of rad/s, the same order of magnitude as the experimental results.

(ii) We have considered an alternative mechanism due to the onset of the tunnelling of a kink wave across the node in the dislocation network. We find a critical angular velocity given by

$$\Omega_D^{crit} = [\Omega_A^{crit} \omega_{TO} 2\pi]^{1/2}. \quad (16)$$

This critical velocity is of the order of 10 rad/s, a little higher than the experimental value. For this mechanism, Ω_D^{crit} is a function of the torsional oscillator frequency whereas this is not true with the other mechanism. We explain this next.

We have investigated this by modelling our calculation along the lines similar to the Josephson effect with the node of the network modelled as the insulating barrier. Under the oscillating rotation, due to the centrifugal force there is an effective "potential" $q\Delta V \approx mL_m^2 L_d \Omega^2/a$ driving the kinks of the dislocations across the node. Ω and hence $q\Delta V$ contains both a DC contribution $q\Delta V_D \approx mL_m^2 L_d \Omega_D^2/a$ and an AC part $q\Delta V_A \approx mL_m^2 L_d 2\Omega_D \Omega_A \cos(\omega_A t)/a$. As we learned from the Josephson equations[17], a current of kinks can develop across the node that contains a term given by

$$J = q\Delta V_A \sin \omega_{TO} t \cos(\delta_0 + q\Delta V_D t/\hbar)/(\hbar\omega_{TO}),$$

where δ_0 is a constant phase difference. The critical velocity is reached when a DC component of the current is developed across the junction. This happens when the quantum energy associated with the oscillation frequency $\hbar\omega_{TO}$ is equal to the effective potential applied due to the centrifugal force $q\Delta V_D$. We obtain a critical DC angular frequency given by eq. (16). We close this paper with other issues that we have considered.

As is mentioned above, in general, the axis of rotation is not parallel to the axis of the dislocation. The crystal orientation can be specified by two Euler angles (θ, Φ) with respect to the rotation axis. (The third angle corresponds to the angle of rotation). The actual displacement from the dislocation motion which contributes to the kinetic energy of the

particles should be $\mathbf{R}\mathbf{u}(\mathbf{R}^{-1}\mathbf{r})$ where \mathbf{R} is the rotation matrix that can be specified by the Euler angles. We have explicitly computed this quantity and verified that our results are as expected. More precisely we find that $\int d^2r \mathbf{r} \times \mathbf{R} \partial_{\mathbf{x}'} \mathbf{u}(\mathbf{R}^{-1}\mathbf{r}) = 0.5 \cos 2\Phi F(\theta)$, $F(\theta) = \int d^2r [(1-2s)(x^4 \cos \theta - y^4 \cos^3 \theta) + (\cos^2 \theta - 1)(3-2s) \cos \theta y^2 x^2] / [(s-1)(\cos^2 \theta y^2 + x^2)^2]$. Similarly we obtain $\int d^2r \mathbf{r} \cdot \mathbf{R} \partial_{\mathbf{x}'} \mathbf{u}(\mathbf{R}^{-1}\mathbf{r}) = \sin(2\Phi) G(\theta)$ where $G = -[x^2 \cos^2(\theta) - y^2] / [y^2 + \cos^2(\theta)x^2]$.

We were also concerned about possible changes in the phonon dispersion due to the rotation and its effect on the energetics of the system. We find that the dominant contribution to the energy change is given by $E_{2sd} = 0.25(\hbar/N) \sum_{k,j} |\mathbf{\Omega} \times \mathbf{e}_j|^2 (2n_{kj} + 1) / \omega_k$. where k, j specifies the wave vector and branch index of the phonons with frequency ω_k , polarization \mathbf{e}_j and occupation number n_{kj} . Since the phonon frequencies are of the order of $10^{12}/\text{sec}$ and Ω is less than rad/s , these corrections are small.

In summary we apply our recently developed model of a Bose condensate of quantum kink wave in solid He4 to understand recent experimental results of the critical velocities and the effect of the steady and oscillating rotations. Estimates of the critical velocities and the change in the Q value of the torsional oscillator with no adjustable parameters are of the same order of magnitude as the experimental results. Their functional dependence on system parameters is discussed. We thank Norbert Mulders for helpful discussions.

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 - [13] There is an additional term $\int d\mathbf{r}\rho[\mathbf{\Omega}_{\mathbf{T}\mathbf{O}}(\mathbf{t}) \times \delta\mathbf{u}]^2$ This term is of the order of $\ln(L_m/a)(\Omega_{TO} \times \delta c)^2$ Since $\Omega_{TO} \approx 10^{-3}/second$, the ratio between the term kept and this term is $10^{-6}\ln(L_m/a)[\delta c/second/v_0]^2$. We take $v_0 \approx 10^7 nm/second$ and $\delta c < 1nm$. This ratio is of the order of 10^{-20} and is much smaller than the first term It will be neglected from now on.
 - [14] $u_x = \arctan(y'/x') + 0.5x'y'/[(x'^2 + y'^2)(1 - \sigma)]$, $u_y = -0.5 [0.5(1 - 2\sigma) \ln(x'^2 + y'^2) + x'^2/(x'^2 + y'^2)]/(1 - \sigma)$ where $x' = x - c_x$, $y' = y - c_y$, σ is the Poisson ratio.
 - [15] See eq. (B1) of ref. ([10])
 - [16] H. Goldstein, "Classical Mechanics", section 4.4
 - [17] See, for example, vol.3 Lectures on Physics. R. P. Feynman, p. 21-16.